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GEODESIC PERPENDICULARS AND EULER CHARACTERISTICS OF PROJECTIVE VARIETIES

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Introduction

In this paper we propose to investigate an old problem: To find the number of perpendiculars drawn from a given point to some submanifold X. Our candidate for X is an n-dimensional algebraic subvariety of complex projective N-space $CP(N)$, and the perpendiculars mean the geodesic lines cutting X orthogonally in $CP(N)$. We shall call these lines geodesic perpendiculars, and answer the above problem in this particular case.

We first assign $+$ or $-$ to every geodesic perpendicular by a method to be explained in §§7 and 8. The number of positively-signed geodesic perpendiculars drawn from a point of $CP(N)$ minus that of negatively-signed ones is called simply the number of geodesic perpendiculars drawn from that point. This number turns out to be a constant on some open dense subset of $CP(N)$, and will be denoted by $n(X)$. Let $\chi(X)$ and $\chi(X \cap H)$ be the Euler characteristics of X and $X \cap H$, a nonsingular hyperplane section; let $T(X)$ and $[-H]$ be the tangent vector bundle and the line bundle associated to a hyperplane section $X \cap H$ respectively. The following triangle of equalities holds:

$$\begin{array}{ccc} & n(x) & \\ /\!\!/ & & \backslash\!\!\backslash \\ \int_X c_n(T(X) \otimes [-H]) & = & \chi(X) - \chi(X \cap H) \end{array}$$

In this paper we will give a proof of the equality of each oblique side of the above triangle, by calculating some curvature integral in a way similar to [5], [7], [12] and by using the Morse theory. As a byproduct we obtain the equality on the base. It is interesting to note that, as to this equality, a much more general formula exists. To be specific, let L be a nonsingular divisor on X. Then we have

$$\chi(X) - \chi(L) = \int_X c_n(T(X) \otimes [-L]),$$

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where $[-L]$ is the line bundle corresponding to the divisor $-L$. This is a consequence of the adjunction formula [2], [10] (e.g., see the formulas (II, 28) in [10, p. 323]). If X is a complete intersection of multi-degree $(d_1 + 1, \cdots, d_{N-n} + 1)$, we can easily calculate the integral of $c_n(T(X) \otimes [-H])$ in the following way.

Let ω be the Kähler form on X, and set

$$\frac{1}{(1 + d_1\omega) \cdots (1 + d_{N-n}\omega)(1 - \omega)} = 1 + m_1\omega + \cdots + m_n\omega^n + \cdots .$$

Then

$$\int_X c_n(T(X) \otimes [-H]) = m_n \times \text{degree of } X.$$

The absolute number of geodesic perpendiculars drawn from a generic point is of course generally greater than $|n(X)|$, but in some cases they can be expected to be the same. Take the example of a complex quadric X with even n. Then we can show that $n(X)$ is the absolute number and equals 2 (see §12).

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1. Complex projective space

Let π be the natural projection of $\mathbf{C}^{N+1} - 0$ to complex projective N-space $CP(N)$. The restriction of π to the unit sphere S^{2N+1} in $\mathbf{C}^{N+1}$ will be denoted by π_S, or briefly by π if there is no fear of confusion. Then $(S^{2N+1}, \pi_S, CP(N))$ is a circle bundle. For $z \in CP(N)$ the coordinates $z_0, \cdots, z_N$ of $\tilde{z} \in \pi^{-1}(z)$ are homogeneous coordinates of z, and are especially called normal coordinates if $\tilde{z} \in S^{N+1}$. Consider the holomorphic map π_* between the tangent vector bundles $T(\mathbf{C}^{N+1} - 0)$ and $T(CP(N))$.

From now on we consider $\mathbf{C}^{N+1}$ as a hermitian space with the inner product: $(\tilde{z}, \tilde{w}) = z_0\bar{w}_0 + \cdots + z_N\bar{w}_N$ where $\tilde{z}, \tilde{w} \in \mathbf{C}^{N+1}$, $\tilde{z} = (z_0, \cdots, z_N)$ and $\tilde{w} = (w_0, \cdots, w_N)$. Let $\tilde{z} \in S^{2N+1}$ and $z = \pi(\tilde{z})$, and denote the orthogonal complement of $\mathbf{C}\tilde{z}$ in $\mathbf{C}^{N+1}$ by M_z. Further we write $\mathfrak{M}_{\tilde{z}}$ for the complex hyperplane through $\tilde{z}$ and parallel to M_z. Then

$$\mathfrak{M} = \bigcup_{\tilde{z} \in S^{2N+1}} (\tilde{z}, \mathfrak{M}_{\tilde{z}})$$

can be viewed as a vector subbundle of $T(S^{2N+1})$. In fact we have

$$T(S^{2N+1}) \simeq M \oplus \mathfrak{R},$$

where $\mathfrak{R}$ on the right side expresses the product bundle with typical fiber **R**. On the other hand π_* gives a vector space isomorphism of each fiber $\mathfrak{M}_{\bar{z}}$ onto $T_z(CP(N))$. Using this isomorphism, we can define a hermitian metric in $T_z(CP(N))$ so that $\pi_* \mid \mathfrak{M}_{\bar{z}}$ becomes an isometry. The metric on $CP(N)$ thus obtained is the Fubini-Study metric

$$ds^2 = \frac{\sum_{A=0}^{N} dz_A d\bar{z}_A}{\sum_{A=0}^{N} z_A \bar{z}_A} - \frac{\sum_{A,B=0}^{N} (\bar{z}_A dz_A)(z_B d\bar{z}_B)}{\left(\sum_{A=0}^{N} z_A \bar{z}_A\right)^2}.$$

Then in normal homogeneous coordinates, the corresponding volume form turns out to be [3, p. 289]

$$(1) \quad dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_N \wedge d\bar{z}_N + \cdots + dz_0 \wedge d\bar{z}_0 \wedge \cdots \wedge dz_{N-1} \wedge d\bar{z}_{N-1}.$$

Let B^{2N} be the unit ball in hermitian space $\mathbf{C}^N$. Then the map which sends $(z_1,\cdots,z_N)$ to the point of $CP(N)$ with homogeneous coordinates $1 - \sqrt{z_1\bar{z}_1 +,\cdots + z_N\bar{z}_N}\ z_1,\cdots,z_N$ is volume-form preserving.

2. Gauss maps

We write $G(m, N+1)$ for the Grassmannian of m-planes (through the origin o) in $\mathbf{C}^{N+1}$. The tautological vector bundle over $G(m, N+1)$ will be denoted by $\mathfrak{S}(m, N+1)$, and the corresponding projectivized vector bundle by $P\mathfrak{S}(m, N+1)$. Let $x \in G(m, N+1)$, and let $y \in \mathfrak{S}(m, N+1)$ be a point lying over x. Then y can be regarded as a complex line contained in m-plane x, and further as one in $\mathbf{C}^{N+1}$, i.e., a point of $CP(N)$. Thus we can get a holomorphic map of $P\mathfrak{S}(m, N+1)$ into $CP(N)$. We may call this map the tautological Gauss map, and denote it by G.

Let X be a differentiable manifold of real dimension $2n$. Consider a differentiable map ϕ of X into $G(m, N+1)$. Then we have a bundle map $\tilde{\phi}$ of the induced bundle $\phi^*(P\mathfrak{S}(m, N+1))$ to $P\mathfrak{S}(m, N+1)$. We denote by G_ϕ, the composition of G and $\tilde{\phi}$, and called it the Gauss map associated to ϕ. Now suppose that $m = N - n + 1$, and that G_ϕ be surjective. Then we can propound a problem of Gauss-Bonnet type. An interesting example will be presented in what follows.

3. The Gauss map G_ϕ considered in this paper

Let X be a nonsingular algebraic subvariety of $CP(N)$, and n the complex dimension of X. The tangent vector space $T_z(X)$ at z of X can be considered as a subspace of $T_z(CP(N))$, the tangent vector space at z of $CP(N)$. Let $\mathfrak{N}_z$ be the orthogonal complement of $T_z(X)$ in $T_z(CP(N))$ with respect to the Fubini-Study metric given in §1. Then there exists one and only one $(N-n)$-dimensional linear subspace of $CP(N)$ which passes through z and is tangent to $\mathfrak{N}_z$ at z. We denote it by $\mathfrak{R}_z$.

We can find a finite collection of homogeneous polynomials $f_i \in C[z_0,\cdots,z_N]$, $1 \leqslant i \leqslant l$, such that the underlying set of X is constituted by all roots of f_i, $1 \leqslant i \leqslant l$. Then $\mathfrak{R}_z$ consists of the complex lines which lie in the $(N-n+1)$-plane $\langle \text{grad } f_1,\cdots,\text{grad } f_l, z\rangle \subset \mathbf{C}^{N+1}$, where

$$\text{grad } f_i = \left(\frac{\partial \bar{f}_i}{\partial z_0},\cdots,\frac{\partial \bar{f}_i}{\partial z_N}\right) \quad (i=1,\cdots,l),$$

and $\langle\ \cdots\ \rangle$ denotes the plane spanned by "$\cdots$". In this way we get a map of X to $G(N-n+1, N+1)$:

$$z \mapsto \phi(z) = \langle \text{grad } f_1,\cdots,\text{grad } f_l, z\rangle.$$

The fiber over $\phi(z)$ of $P\mathfrak{S}(N-n+1, N+1)$ is exactly $\mathfrak{R}_z$. We write the induced bundle $\phi^*(P\mathfrak{S}(N-n+1, N+1))$ as $\mathfrak{R}$. $\mathfrak{R}$ has a natural almost complex structure which is not necessarily integrable. The Gauss map associated to ϕ sends $\mathfrak{R}$ to the complex projective space of the same dimension differentiably.

Let us introduce inhomogeneous coordinates into $CP(N)$ by

$$x_1 = \frac{z_1}{z_0},\cdots,x_N = \frac{z_N}{z_0}, \tag{2}$$

where z is supposed to be a point of $U_0 = \{z \in CP(N) \mid z_0 \neq 0\}$. Consider a complex quadric given by

$$z_0^2 + \cdots + z_N^2 = 0 \tag{3}$$

as X, and use (2) to express G_ϕ by

$$w_1 = \frac{x_1 + x_0\bar{x}_1}{1+x_0},\cdots,w_N = \frac{x_N + x_0\bar{x}_N}{1+x_0},$$

where $w_1,\cdots,w_N$ are inhomogeneous coordinate on the image space, and x_0 is an inhomogeneous coordinate on $CP(1)$. We therefore have

$$dw_i = \frac{(\bar{x}_i - x_i)dx_0}{(1+x_0)^2} + \frac{dx_i}{1+x_0} + \frac{x_0 d\bar{x}_i}{1+x_0},$$

where $i = 1,\cdots,N$. From (3) it follows that

$$x_1^2 = -x_2^2 - \cdots - x_N^2 - 1.$$

We can view $x_0, x_2, \cdots, x_N$ as independent variables. On the other hand

$$dx_1 = \frac{1}{x_1}(-x_2 dx_2 - \cdots - x_N dx_N).$$

Hence we can write

$$(*)\qquad \begin{aligned} &dw_1 \wedge d\bar{w}_1 \wedge \cdots \wedge dw_N \wedge d\bar{w}_N \\ &\quad = J dx_0 \wedge d\bar{x}_0 \wedge dx_2 \wedge d\bar{x}_2 \wedge \cdots \wedge dx_N \wedge d\bar{x}_N, \end{aligned}$$

where J is the Jacobian determinant which we calculate below. First we see

$$\begin{aligned} dw_i \wedge d\bar{w}_i = {} & \frac{|x_i - \bar{x}_i|^2}{|1 + x_0|^4} dx_0 \wedge d\bar{x}_0 + \frac{(1 - |x_0|^2)}{|1 + x_0|^2} dx_i \wedge d\bar{x}_i \\ & + \frac{(\bar{x}_i - x_i)}{(1 + x_0)|1 + x_0|^2} dx_0 \wedge (d\bar{x}_i + \bar{x}_0 dx_i) \\ & + \frac{(x_i - \bar{x}_i)}{(1 + x_0)|1 + x_0|^2} (dx_i + x_0 d\bar{x}_i) \wedge d\bar{x}_0. \end{aligned}$$

The factor $dx_0 \wedge d\bar{x}_0$ appears in two ways, as the first term in the above $dw_i \wedge d\bar{w}_i$ and as

$$\frac{(1 - |x_0|^2)(x_i - \bar{x}_i)(\bar{x}_j - x_j)}{|1 + x_0|^6} dx_0 \wedge d\bar{x}_0 \wedge (d\bar{x}_i \wedge dx_j - dx_i \wedge d\bar{x}_j),$$

where $i \neq j$. Hence the expansion of the left side of the above equation $(*)$ decomposes into three parts:

The first part is

$$A \sum_{i=1}^{N} |x_i|^2 |x_i - \bar{x}_i|^2 dx_0 \wedge d\bar{x}_0 \wedge dx_2 \wedge d\bar{x}_2 \wedge \cdots \wedge dx_N \wedge d\bar{x}_N,$$

the second is

$$\begin{aligned} &A \sum_{1<i<j} (x_i \bar{x}_j + \bar{x}_i x_j)(x_i - \bar{x}_i)(\bar{x}_j - x_j) \\ &\qquad \cdot dx_0 \wedge d\bar{x}_0 \wedge dx_2 \wedge d\bar{x}_2 \wedge \cdots \wedge dx_N \wedge d\bar{x}_N, \end{aligned}$$

and the last one is

$$\begin{aligned} &A(x_1 - \bar{x}_1) \sum_{1<i} (\bar{x}_i - x_i)(x_1 \bar{x}_i + x_i \bar{x}_1) \\ &\qquad \cdot dx_0 \wedge d\bar{x}_0 \wedge dx_2 \wedge d\bar{x}_2 \wedge \cdots \wedge dx_N \wedge d\bar{x}_N. \end{aligned}$$

By putting

$$A = \frac{\left(1 - |x_0|^2\right)^{N-1}}{|1 + x_0|^{2N+2}|x_1|^2},$$

we obtain

$$\begin{aligned} J &= \text{the jacobian of the Gauss map of quadric (3)} \\ &= A \sum_{i,j=1}^{N} x_i(\bar{x}_i - x_i)\bar{x}_j(x_j - \bar{x}_j) \\ &= \frac{\left(1 - |x_0|^2\right)^{N-1}}{|1 + x_0|^{2N+2}|x_1|^2}\left(1 + \sum x_i\bar{x}_i\right)^2. \end{aligned}$$

Now let us go back to a nonsingular n-dimensional projective variety $X \subset CP(N)$. We choose a unitary frame $(e_1,\cdots,e_n)$ over an open $U \subset X$ for $T(X)$. We can find $\tilde{e}_1,\cdots,\tilde{e}_n \in \mathfrak{M}$ in a unique way such that $\pi_*(\tilde{e}_i) = e_i$, $i = 1,\cdots,n$. Let us denote by $\mathfrak{S}_{\tilde{z}} \subset \mathfrak{M}_{\tilde{z}}$ the subspace which is spanned by $\tilde{e}_1,\cdots,\tilde{e}_n$ at $\tilde{z}$. Put

$$\tilde{\mathfrak{S}} = \bigcup \mathfrak{S}_{\tilde{z}}\left(\tilde{z} \in \pi^{-1}(X)\right).$$

Then $\tilde{\mathfrak{S}}$ is a vector bundle over $\pi^{-1}(X)$, isomorphic to the pull-back of $T(X)$, and $\tilde{e}_1,\cdots,\tilde{e}_n$ form a frame for $\tilde{\mathfrak{S}}$. We extend the frame to a unitary frame $(\tilde{e}_1,\cdots,\tilde{e}_N)$ for $\mathfrak{M} \mid \pi^{-1}(X)$. Then

$$\tilde{e}_0 = \tilde{z}, \tilde{e}_1,\cdots,\tilde{e}_N$$

form a unitary frame of product bundle $\mathcal{C}^{N+1}$ with $\mathbf{C}^{N+1}$ as typical fiber over $\pi^{-1}(X)$. Taking a local section σ of $(S^{2N+2}, \pi, CP(N))$ over U, we consider $\tilde{e}_0,\cdots,\tilde{e}_N$ as vector-valued differentiable functions defined over U. On each fiber $\mathfrak{N}_z$ of the bundle $\mathfrak{N}$ in §3, we introduce normal homogeneous coordinates $u_0, u_{n+1},\cdots,u_N$ with respect to $\tilde{e}_0, \tilde{e}_{n+1},\cdots,\tilde{e}_N$. Obviously $u_0, u_{n+1},\cdots,u_N$ can be also regarded as normal coordinates of point u of $CP(N-n)$. The map defined by

$$(z, u) \mapsto u_0\tilde{e}_0 + u_{n+1}\tilde{e}_{n+1} + \cdots + u_N\tilde{e}_N$$

gives an isomorphism between $U \times CP(N-n)$ and $\mathfrak{N} \mid U$. Up to this isomorphism, the Gauss map G_ϕ can be expressed by

$$(z, u) \mapsto \pi\left(u_0\tilde{e}_0 + u_{n+1}\tilde{e}_{n+1} + \cdots + u_N\tilde{e}_N\right).$$

4. A connection

For fixed $z \in X$, $\mathfrak{S}_{\tilde{z}}$ are parallel to one another. Denote by $\mathfrak{S}_z$ the n-dimensional linear space through the origin which is parallel to $\mathfrak{S}_{\tilde{z}}$. Then we can define a map of X to $G(n, N+1)$ by $z \mapsto \mathfrak{S}_z$. Denote by $\mathfrak{S}$ the pull-back by this map of tautological vector bundle $\mathfrak{S}(n, N+1)$ over $G(n, N+1)$. We see easily that the vector bundle $\mathfrak{S}$ over X is isomorphic to $T(X) \otimes [-H]$. Introduce a connection in this bundle by orthogonal projection as follows [6].

First we write

$$d\tilde{e}_A = \sum_B \omega_{AB}\tilde{e}_B,$$

where A, B range over $0, 1, \cdots, N$. Then

$$\omega_{AB} + \overline{\omega}_{BA} = 0, \quad \omega_{0,n+1} = \cdots = \omega_{0,N} = 0.$$

In what follows, let letters $r, s, \cdots$ run through $n+1, \cdots, N$, and $i, j, \cdots$ through $1, \cdots, n$. Now we would like to make a change in notation. Write Ω_{ir} instead of ω_{ir}, ω_i instead of ω_{i0}, and ω_0 instead of ω_{00}. Then

$$\begin{aligned} d\tilde{e}_0 &= \sum_j \omega_j \tilde{e}_j + \omega_0 \tilde{e}_0, \\ d\tilde{e}_i &= \sum_j \omega_{ij}\tilde{e}_j + \sum_r \Omega_{ir}\tilde{e}_r - \overline{\omega}_i \tilde{e}_0, \\ d\tilde{e}_r &= -\sum_j \overline{\Omega}_{jr}\tilde{e}_j + \sum_s \omega_{rs}\tilde{e}_s. \end{aligned}$$

The matrix form (ω_{ij}) gives a connection on $\mathfrak{S}$, and the curvature forms θ_{ij} are defined by

$$\theta_{ij} = d\omega_{ij} + \sum_k \omega_{ik} \wedge \omega_{kj}.$$

Then

$$\theta_{ij} = -\sum_{r^*} \Omega_{i\gamma^*} \wedge \Omega_{\gamma^* j} = \sum_r \Omega_{i\gamma^*} \wedge \overline{\Omega}_{j\gamma^*},$$

where γ^* runs through $0, n+1, \cdots, N$. We denote the curvature matrix by θ. It is interesting to note that θ does not depend on the choice of section σ. In fact let $\lambda\sigma$ be the second section of $(S^{2N+1}, \pi, CP(N))$. We write θ' for the corresponding curvature matrix. Then

$$\begin{aligned} \theta' &= d(\omega + d(\log\lambda)1_n) + (\omega + d(\log\lambda)1_n) \wedge (\omega + d(\log\lambda)1_n) \\ &= \theta + d(\log\lambda) \wedge \omega + \omega \wedge d(\log\lambda) = \theta. \end{aligned}$$

5. Volume form dv_N

Remember that the Gauss map G_ϕ sends N into $CP(N)$. Let us rewrite the volume form dv_N of $CP(N)$ in the following way:

$$dv_N = (-1)^{N(N-1)/2}\frac{1}{N!}\cdot\frac{\sqrt{-1}}{2}\sum dz_{A_1}\wedge\cdots\wedge dz_{A_N}\wedge d\bar{z}_{A_1}\wedge\cdots\wedge d\bar{z}_{A_N},$$

where $A_1,\cdots,A_N = 0,\cdots,N$, and we use normal homogeneous coordinates. We begin with the calculation of $G_\phi^*(dv_N)$. Write

$$\omega_{A_1,\cdots,A_N} = dz_{A_1}\wedge\cdots\wedge dz_{A_N}.$$

Let us consider n linearly independent infinitesimal vectors $dz, \delta z, \cdots$ on $U \subset X$ and $(N-n)$ linearly independent infinitesimal vectors $d'u, \delta'u, \cdots$ on $CP(N-n)$. We identify $dz, \delta z, \cdots$ with $(dz, 0), (\delta z, 0), \cdots$, and $d'u, \delta'u, \cdots$ with $(0, d'u), (0, \delta'u), \cdots$ respectively. Gauss map G_ϕ sends them to $T_z(CP(N))$; they are given by

$$\pi_*\Big(u_0 d\tilde{e}_0 + \sum_r u_r d\tilde{e}_r\Big),$$

$$\pi_*\Big(u_0 \delta\tilde{e}_0 + \sum_r u_r \delta\tilde{e}_r\Big),$$

$$\cdots\cdots\cdots$$

$$\pi_*\Big(d'u_0 \tilde{e}_0 + \sum_r d'u_r \tilde{e}_r\Big),$$

$$\pi_*\Big(\delta'u_0 \tilde{e}_0 + \sum_r \delta'u_r \tilde{e}_r\Big),$$

$$\cdots\cdots\cdots$$

On the other hand we have

$$u_0 d\tilde{e}_0 + \sum_r u_r d\tilde{e}_r = u_0\omega_0\tilde{e}_0 - \sum_i\Big(u_0\bar{\Omega}_{i0} + \sum_r u_r\bar{\Omega}_{ir}\Big)\tilde{e}_i + \sum_{r,s} u_r\omega_{rs}\tilde{e}_s.$$

Note that any unitary transformation in $\mathbf{C}^{N+1}$ leaves the volume form (1) invariant. Hence we can take $\tilde{e}_1,\cdots,\tilde{e}_n,\ \tilde{e}_0,\ \tilde{e}_{n+1},\cdots,\tilde{e}_N$ as the base of $\mathbf{C}^{N+1}$ without any change in (1). Consider the matrix

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} & a_{1,n+1} & a_{1,n+2} & \cdots & a_{1,N+1} \\ a_{21} & \cdots & a_{2n} & a_{2,n+1} & a_{2,n+2} & \cdots & a_{2,N+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} & a_{n,n+1} & a_{n,n+2} & \cdots & a_{n,N+1} \\ 0 & \cdots & 0 & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

where

$$a_{11} = -\left(u_0\overline{\Omega}_{10}(dz) + \sum_r u_r\overline{\Omega}_{1r}(dz)\right),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_{1n} = -\left(u_0\overline{\Omega}_{n0}(dz) + \sum_r u_r\overline{\Omega}_{nr}(dz)\right),$$

$$a_{21} = -\left(u_0\overline{\Omega}_{10}(\delta z) + \sum_r u_r\overline{\Omega}_{1r}(\delta z)\right),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_{2n} = -\left(u_0\overline{\Omega}_{n0}(\delta z) + \sum_r u_r\overline{\Omega}_{nr}(\delta z)\right),$$

$$a_{1,n+1} = u_0\omega_0(dz),\ a_{2,n+1} = u_0\omega_0(\delta z), \cdots,$$

$$a_{n+1,n+1} = du_0,\ a_{n+2,n+1} = \delta u_0, \cdots,$$

$$a_{1,s+1} = \sum_r u_r\omega_{rs}(dz),\ a_{2,s+1} = \sum_r u_r\omega_{rs}(\delta z), \cdots,$$

$$a_{n+1,s+1} = du_s,\ a_{n+2,s+1} = \delta u_s, \cdots, \qquad (s = n+1, \cdots, N).$$

Then $\omega_{A_1,\cdots,A_N}$ equals the minor of type $(1,\cdots,N;\ A_1+1,\cdots,A_N+1)$ of this matrix. On the other hand, dv_N remains invariant even if we replace z by λz such that $\lambda \in \mathbf{C}$ and $|\lambda| = 1$, [3]. Therefore we can assume, for a while, that each line vector of the matrix is orthogonal to $\tilde{u}' = (0,\cdots,0,u_0,u_{n+1},\cdots,u_N) \in \mathbf{C}^{N+1}$ at the point under consideration. Hence we see that

the k-th component of $\tilde{u}' = (-1)^{k-1}$ the minor of
type $(1,\cdots,N;\ 1,\cdots,\hat{k},\cdots,N+1)$ (up to a common factor),

where the roof over the letter k means that the letter is to be omitted. From this fact it follows especially that if $\{A_1,\cdots,A_N\} \not\supset \{0,\cdots,n-1\}$, then $\omega_{A_1,\cdots,A_N}$ vanish. Hence we have

$$dv_N = \left(\frac{\sqrt{-1}}{2}\right)^n \left(u_0\overline{\Omega}_{10} + \sum_r u_r\overline{\Omega}_{1r}\right) \wedge \left(\bar{u}_0\Omega_{10} + \sum_r \bar{u}_r\Omega_{1r}\right) \wedge \cdots$$
$$\wedge \left(u_0\overline{\Omega}_{n0} + \sum_r u_r\overline{\Omega}_{nr}\right) \wedge \left(\bar{u}_0\Omega_{n0} + \sum_r \bar{u}_r\Omega_{nr}\right) \wedge dv_{N-n}.$$

Thus there are integrals of the form:

$$\int_{CP(N-n)} u_0^{\alpha_0}\bar{u}_0^{\beta_0}u_{n+1}^{\alpha_1}\bar{u}_{n+1}^{\beta_1}\cdots u_N^{\alpha_{N-n}}\bar{u}_N^{\beta_{N-n}}dv_{N-n},$$

where $\alpha_0,\cdots,\alpha_{N-n}$ and $\beta_0,\cdots,\beta_{N-n}$ are nonnegative integers. If we suppose $u_0 \in \mathbf{R}$ and $u_0 > 0$, then the above integrals become

$$\int_{B^{N-n}}\left(1-\sum_r u_r\bar{u}_r\right)^{\alpha_0+\beta_0} u_{n+1}{}^{\alpha_1}\bar{u}_{n+1}{}^{\beta_1}\cdots u_N{}^{\alpha_{N-n}}\bar{u}_N{}^{\beta_{N-n}} \cdot du_{n+1}\wedge d\bar{u}_{n+1}\wedge\cdots\wedge du_N\wedge d\bar{u}_N, \tag{4}$$

where $B^{N-n}=\{(u_{n+1},\cdots,u_N)\in \mathbf{C}^{N-n}\,|\,u_{n+1}\bar{u}_{n+1}+\cdots+u_N\bar{u}_N\leqslant 1\}$. Then the integrals

$$\int_{|\lambda|\leqslant\beta} F\lambda^{\alpha}d\lambda\wedge d\bar{\lambda}$$

vanishes for a strictly positive integer α. Hence the integrals (4) must vanish unless $\alpha_0=\beta_0,\cdots,\alpha_{N-n}=\beta_{N-n}$. We therefore find

$$\int_N dv_N=\left(\frac{\sqrt{-1}}{2}\right)^n\int\sum\sigma(r_1,\cdots,r_n)\bar{\Omega}_{1,s_1}\wedge\Omega_{r_1,s}\wedge\cdots \wedge\bar{\Omega}_{n,s_n}\wedge\Omega_{r_n,s_n}\int_{CP(N-n)}u_{s_1}\bar{u}_{s_1}\cdots u_{s_n}\bar{u}_{s_n}dv_{N-n}, \tag{5}$$

where $\sigma(r_1,\cdots,r_n)$ is the signature of the permutation $(r_1,\cdots,r_n)$, and the meaning of the summation is a little complicated, though it is clear from the context. But after the calculation is made in the next section, this summation will be replaced by a simple one.

6. Calculation of a Dirichlet's integral

Let $u_0,\cdots,u_m$ be normal homogeneous coordinates of $u\in CP(m)$, and $\alpha_0,\cdots,\alpha_m$ arbitrary positive real numbers. Then we have

Lemma.

$$\int_{CP(m)}(u_0\bar{u}_0)^{\alpha_0-1}\cdots(u_m\bar{u}_m)^{\alpha_m-1}dv_m=\pi^m\frac{\Gamma(\alpha_0)\cdots\Gamma(\alpha_m)}{\Gamma(\alpha_0+\cdots+\alpha_m)}.$$

Proof. Let $f(\tau)$ be a continuous function of one real variable running through [0, 1]. Then, according to [13],

$$\int\int\cdots\int f(t_1+\cdots+t_m)t_1{}^{\alpha_1-1}\cdots t_m{}^{\alpha_m-1}dt_1\cdots dt_m =\frac{\Gamma(\alpha_1)\cdots\Gamma(\alpha_m)}{\Gamma(\alpha_1+\cdots+\alpha_m)}\int_0^1 f(\tau)\tau^{\alpha_1+\cdots+\alpha_m-1}d\tau. \tag{6}$$

Let us denote by I the integral on the left side of (6). Then

$$I = \left(\frac{\sqrt{-1}}{2}\right)^m \int_{B^m} (1 - u_1\bar{u}_1 - \cdots - u_m\bar{u}_m)^{\alpha_0 - 1}(u_1\bar{u}_1)^{\alpha_1 - 1} \cdots$$

$$\cdot (u_m\bar{u}_m)^{\alpha_m - 1} du_1 \wedge d\bar{u}_1 \wedge \cdots \wedge du_m \wedge d\bar{u}_m$$

$$= 2^{2m} \underset{\substack{t_1^2 + \cdots + t_{2m}^2 \leqslant 1 \\ t_1, \cdots, t_{2m} \geqslant 0}}{\int\int \cdots \int} (1 - t_1^2 - \cdots - t_{2m}^2)^{\alpha_0 - 1}(t_1^2 + t_2^2)^{\alpha_1 - 1} \cdots$$

$$\cdot (t_{2m-1}^2 + t_{2m}^2)^{\alpha_m - 1} dt_1 \wedge \cdots \wedge dt_{2m},$$

where we have put $u_i = t_{2i-1} + \sqrt{-1}\, t_{2i}$ with t_{2i-1}, t_{2i} reals $(i = 1, \cdots, m)$. Suppose $\alpha_1, \cdots, \alpha_m$ be integers $\geqslant 1$ (still α_0 is arbitrary), and expand the factors $(t_{2i-1}^2 + t_{2i}^2)^s$, $(s = \alpha_i - 1)$. Then I becomes a sum of Dirichlet's integrals of type (6). Each term of the sum has a common factor

$$\int_0^1 (1 - \tau)^{\alpha_0 - 1} \tau^{\alpha_1 + \cdots + \alpha_m - 1} d\tau (= B(\alpha_1 + \cdots + \alpha_m, \alpha_0)),$$

and other factors of each term do not contain α_0. Hence we can write I in the form:

$$\frac{\Gamma(\alpha_0) \times \text{a factor not depending on } \alpha_0}{\Gamma(\alpha_0 + \cdots + \alpha_m)}.$$

Let us consider I as a function of real variables $\alpha_0, \cdots, \alpha_m > 0$ again. Since I is symmetric with respect to these variables, we can write

$$I = c \frac{\Gamma(\alpha_0) \cdots \Gamma(\alpha_m)}{\Gamma(\alpha_0 + \cdots + \alpha_m)},$$

where c is a constant. We can determine c by setting $\alpha_0 = \cdots = \alpha_m = 1$. In fact, we get $c = \pi^m$. This completes the proof of the lemma.

Going back to (5) and using the above lemma, we can rewrite the right side of (5) in the form:

$$(7) \quad \frac{\pi^{N-n}}{N!} \left(\frac{\sqrt{-1}}{2}\right)^n \int_X \sum_{(r_1, \cdots, r_n)} \sigma(r_1, \cdots, r_n) \sum_{s_1^*, \cdots, s_n^*} \bar{\Omega}_{1,s_n} \wedge \Omega_{r_1, s_1} \wedge \cdots \wedge \bar{\Omega}_{n,s_n} \wedge \Omega_{r_n, s_n}$$

$$= \frac{\pi^N}{N!} \int_X c_n(\mathfrak{S}),$$

where $c_n(\mathfrak{S})$ is the highest Chern class of the vector bundle $\mathfrak{S}$ defined in §3, the first summation ranges over all the permutations of $1, \cdots, n$, and $s_i^* (i = 1, \cdots, n)$ run through $0, n + 1, \cdots, N$.

7. Geodesic perpendiculars

Let $z, w \in CP(N)$ $(z \neq w)$. Let $z_0, \cdots, z_N$ and $w_0, \cdots, w_N$ be respective normal homogeneous coordinates, and write

$$\tilde{w}' = \frac{\tilde{w} - (\tilde{w}, \tilde{z})\tilde{z}}{|\tilde{w} - (\tilde{w}, \tilde{z})\tilde{z}|} = \frac{\tilde{w} - (\tilde{w}, \tilde{z})\tilde{z}}{\sqrt{1 - (\tilde{w}, \tilde{z})(\tilde{w}, \tilde{z})}}.$$

We may assume that $(\tilde{w}, \tilde{z}) \in R$ and $(\tilde{w}, \tilde{z}) \geqslant 0$. Then we can find an angle θ_0 $(0 \leqslant \theta_0 \leqslant \pi/2)$ such that

$$\cos\theta_0 = (\tilde{w}, \tilde{z}), \quad \sin\theta_0 = \sqrt{1 - (\tilde{w}, \tilde{z})(\tilde{w}, \tilde{z})}.$$

Define a map $\iota\colon CP(1) \to CP(N)$ by $\iota(u) = \pi_*(u_0\tilde{z} + u_1\tilde{w}')$ where u_0, u_1 are normal homogeneous coordinates of $u \in CP(1)$. Then we have

$$\iota(\cos\theta, \sin\theta) = \begin{cases} z & \text{for } \theta = 0, \\ w & \text{for } \theta = \theta_0. \end{cases}$$

We can see that ι is an isometry, $\theta \mapsto (\cos\theta, \sin\theta)$ is a geodesic on $CP(1)$ with arc length θ, and $\iota(CP(1))$ is totally geodesic in $CP(N)$. Hence

$$\theta \mapsto (\cos\theta, \sin\theta) = \cos\theta\, z + \sin\theta\, w'$$

is a geodesic joining z with w. Therefore the distance $\delta(z, w)$ between z and w is given by $\cos\delta(z, w) = (\tilde{z}, \tilde{w})$. If we replace $(\tilde{z}, \tilde{w})$ by $|(\tilde{z}, \tilde{w})|$, we obtain the expression of $\delta(z, w)$, which does not depend on the special choice of normal homogeneous coordinates. Thus $\cos\delta(z, w) = |(\tilde{z}, \tilde{w})|$ where $0 \leqslant \delta(z, w) \leqslant \pi/2$.

Let $w \in CP(N) - X$ and $z \in X$. The unit tangent vector of the geodesic joining z to w is $\pi_*(w')$, which is orthogonal to X if and only if $\tilde{w} \in \langle \tilde{e}_{n+1}, \cdots, \tilde{e}_N, \tilde{z} \rangle$. In terms of the Gauss map, this means that we can draw from w a geodesic cutting x orthogonally if and only if w belongs to the image of G_ϕ. We call such geodesics "geodesic perpendiculars from w". Suppose any foot point z of geodesic perpendiculars from w be not conjugate to w in $CP(N)$. Then the absolute number of geodesic perpendiculars from w is the cardinality of $G_\phi^{-1}(w)$, that is, G_ϕ^{-1} is in 1-1 correspondence with the set of geodesic perpendiculars from w. Let $y \in G_\phi^{-1}(w)$. Then the geodesic perpendicular corresponding to y is said to be positive or negative according as the Jacobian of G_ϕ at y is > 0 or < 0. We define the number of geodesic perpendiculars from w to be the number of positive ones minus that of negative ones.

From now on we do not assume homogeneous coordinates $z_0, \cdots, z_N$ be normal. We introduce local coordinates $x_1, \cdots, x_n$ in X, and consider $z_0, \cdots, z_N$

as holomorphic functions of $x_1, \cdots, x_n$. Set

$$h(z, \bar{z}) = \frac{(\tilde{z}, \tilde{w})\overline{(\tilde{z}, \tilde{w})}}{(\tilde{z}, \tilde{z})} \quad \text{for } z \in X, w \in CP(N).$$

We restrict h on X and view it as a function on X in what follows. First we have

$$\frac{\partial}{\partial x_i} h(z, \bar{z}) = \frac{\overline{(\tilde{z}, \tilde{w})}}{(\tilde{z}, \tilde{z})} \left(\frac{\partial \tilde{z}}{\partial x_i}, \tilde{w} - \frac{(\tilde{w}, \tilde{z})}{(\tilde{z}, \tilde{z})} \tilde{z} \right), \quad i = 1, \cdots, n. \tag{8}$$

Since

$$\frac{\partial}{\partial \bar{x}_i} h(z, \bar{z}) = 0 \Leftrightarrow \frac{\partial}{\partial x_i} h(z, \bar{z}) = 0 \ (i = 1, \cdots, n),$$

$h(z, \bar{z})$ has a critical point at z if and only if w belongs to

$$\operatorname{Im} G_\phi \cup \{v \in CP(N) \mid (\tilde{v}, \tilde{z}) = 0\}.$$

Suppose that X is in general position, namely, that there is no hyperplane containing X. Then h takes a positive value at some point on X, and the maximum points of h belong to $\operatorname{Im} G_\phi$. Thus we have

Proposition. *If X is in general position in $CP(N)$, the Gauss map G_ϕ is surjective. In other words, we can draw at least one geodesic perpendicular from any point of $CP(N)$ to X.*

It follows from the surjectivity of G_ϕ that

$$\int_{\mathfrak{N}} dv_N = \text{degree of } G_\phi = \int_{CP(N)} dv_N. \tag{9}$$

From (7) and (9), we obtain

$$\text{degree of } G_\phi = \int_X c_n(\mathfrak{S}).$$

8. The signs of the hessian and the perpendicular

Throughout this section, we consider only generic $w \in CP(N)$. The geodesic perpendiculars from w to

$$X' = X - \{z \in X \mid (\tilde{z}, \tilde{w}) = 0\}$$

is in 1-1 correspondence with the foot points of them. The purpose of this section is to find a relation between the sign of a geodesic perpendicular from w and the sign of the hessian of h at its foot z. By differentiating (8) formally with respect to w_A, $\bar{w}_A$, we have

$$\frac{\partial}{\partial w_A}\frac{\partial}{\partial x_i}h = \frac{\bar{z}_A}{(\tilde{z},\tilde{z})}\left(\frac{\partial \tilde{z}}{\partial x_i}, \tilde{w} - \frac{(\tilde{w},\tilde{z})}{(\tilde{z},\tilde{z})}\tilde{z}\right), \tag{10}$$

$$\frac{\partial}{\partial \bar{w}_A}\frac{\partial}{\partial x_i}h = \frac{(\tilde{w},\tilde{z})}{(\tilde{z},\tilde{z})}\left(\frac{\partial z_A}{\partial x_i} - \frac{\left(\frac{\partial z_A}{\partial x_i}, \tilde{z}\right)}{(\tilde{z},\tilde{z})}\tilde{z}_A\right). \tag{11}$$

Suppose $z \in X'$ be a foot point of a geodesic perpendicular from w. The equality (8) implies that

$$\frac{\partial}{\partial w_A}\frac{\partial}{\partial x_i}h = 0\ (i = 1,\cdots,n;\ A = 0,\cdots,N),$$

so that

$$\sum \frac{\partial^2 h}{\partial x_i \partial x_j}dx_j + \sum \frac{\partial^2 h}{\partial x_i \partial \bar{x}_j}d\bar{x}_j = \sum B_{iA}d\bar{w}_A. \tag{12}$$

On the other hand, we introduce inhomogeneous coordinates

$$x_{n+1} = \frac{u_{n+1}}{u_0},\cdots,x_N = \frac{u_N}{u_0},$$

and the range space $CP(N)$ of G_ϕ respectively.

$$y_1 = \frac{w_1}{w_0},\cdots,y_N = \frac{w_N}{w_0}$$

for the fibers diffeomorphic to $CP(N)$. Since $y_1,\cdots,y_N$ are holomorphic with respect to $x_{n+1},\cdots,X_N$, we can write

$$dy_A = \sum_i \cdots dx_i + \sum_i \cdots d\bar{x}_i + \sum_r \cdots dx_r.$$

Since an infinitesimal vector $(dx_{n+1},\cdots,dx_N)$ is sent to the tangent vector space $T_w(CP(N))$ injectively by the Gauss map G_ϕ, we can solve these equations for $dx_{n+1},\cdots,dx_N$. Hence we have

$$dx_r = \sum_{A=1}^{N} C_{rA}dy_A + \sum_i D_{ri}dx_i + \sum_i D'_{ri}d\bar{x}_i.$$

Denote the matrices

$$\left(\frac{\partial^2 h}{\partial x_i \partial \bar{x}_j}\right), \quad \left(\frac{\partial^2 h}{\partial x_i \partial x_j}\right)$$

by H, H' respectively. Then the hessian of h is equal to the determinant

$$e = \begin{vmatrix} H & H' \\ \overline{H'} & \overline{H} \end{vmatrix}.$$

Note that in (12) we can set $dw_0 = 0$, $dw_1 = dy_1, \cdots, dw_N = dy_N$. Hence we have

$$\begin{aligned} e\, dx_1 \wedge \cdots \wedge dx_N \wedge d\bar{x}_1 \wedge \cdots \wedge d\bar{x}_N & \\ = (-1)^{n(N-n)} e\, dx_1 \wedge \cdots \wedge dx_n \wedge d\bar{x}_1 \wedge \cdots \wedge d\bar{x}_n & \\ \wedge dx_{n+1} \wedge \cdots \wedge dx_N \wedge d\bar{x}_{n+1} \wedge \cdots \wedge d\bar{x}_N & \\ = \left(\sum_{i_1} \frac{\partial^2 h}{\partial x_1 \partial \bar{x}_{i_1}} dx_{i_1} + \sum_{i_1} \frac{\partial^2 h}{\partial x_1 \partial x_{i_1}} d\bar{x}_{i_1} \right) \wedge \cdots \wedge \left(\sum_{i_n} \frac{\partial^2 h}{\partial x_n \partial \bar{x}_{i_n}} dx_{i_n} \right. & \\ \left. + \sum_{i_n} \frac{\partial^2 h}{\partial x_n \partial x_{i_n}} dx_{i_n} \right) \wedge \left(\sum_{j_1} \frac{\partial^2 h}{\partial x_1 \partial x_{j_1}} dx_{j_1} + \sum_{j_1} \frac{\partial^2 h}{\partial x_1 \partial \bar{x}_{j_1}} d\bar{x}_{j_1} \right) \wedge \cdots & \\ \wedge \left(\sum_{j_n} \frac{\partial^2 h}{\partial x_n \partial x_{j_n}} dx_{j_n} + \sum_{j_n} \frac{\partial^2 h}{\partial x_n \partial \bar{x}_{j_n}} d\bar{x}_{j_n} \right) \wedge dx_{n+1} \wedge \cdots \wedge dx_N & \\ \wedge d\bar{x}_{n+1} \wedge \cdots \wedge d\bar{x}_N & \\ = \left(\sum \bar{B}_{1A_1} dy_{A_1} \right) \wedge \cdots \wedge \left(\sum \bar{B}_{nA_n} dy_{A_n} \right) \wedge \left(\sum B_{1A_1'} d\bar{y}_{A_1'} \right) \wedge \cdots & \\ \wedge \left(\sum B_{nA_n'} d\bar{y}_{A_n'} \right) \wedge \left(\sum C_{n+1,B_1} dy_{B_1} \right) \wedge \cdots \wedge \left(\sum C_{NB_{N-n}} dy_{B_{N-n}} \right) & \\ \wedge \cdots \wedge \left(\sum \bar{C}_{n+1,B_1'} d\bar{y}_{B_1'} \right) \wedge \cdots \wedge \left(\sum \bar{C}_{n+1,B_{N-n}'} d\bar{y}_{B_{N-n}'} \right) & \\ = \begin{vmatrix} \bar{B} \\ C \end{vmatrix} \begin{vmatrix} B \\ \bar{C} \end{vmatrix} dy_1 \wedge \cdots \wedge dy_N \wedge d\bar{y}_1 \wedge \cdots \wedge d\bar{y}_N, & \end{aligned}$$

where

$$B = \begin{pmatrix} B_{11} \cdots B_{1N} \\ B_{n1} \cdots B_{nN} \end{pmatrix}, \quad C = \begin{pmatrix} C_{n+1,1} \cdots C_{n+1,N} \\ C_{N1} \cdots C_{NN} \end{pmatrix}.$$

Since

$$\begin{aligned} dy_1 \wedge \cdots \wedge dy_N \wedge d\bar{y}_1 \wedge \cdots \wedge d\bar{y}_N & \\ = \text{the jacobian} \times dx_1 \wedge \cdots \wedge dx_N \wedge d\bar{x}_1 \wedge \cdots \wedge d\bar{x}_N, & \end{aligned}$$

we can get

$$\text{the hessian} = \begin{vmatrix} \bar{B} \\ C \end{vmatrix} \begin{vmatrix} \bar{B} \\ C \end{vmatrix} \times \text{the jacobian}.$$

Thus we can state the following proposition.

Proposition. *Let $z \in X'$ be a nondegenerate critical point of h. Let p be a point over z of the bundle $\mathfrak{N}$ such that the image of p by the Gauss map is just w. Then*

$$\text{the index of } h \text{ at } z \equiv 0 \pmod 2 \text{ if the jacobian at } p > 0,$$
$$\text{the index of } h \text{ at } z \equiv 1 \pmod 2 \text{ if the jacobian at } p < 0,$$

where the jacobian means that of the Gauss map G_ϕ.

We write

$$h_w(z) = \frac{1}{(\tilde{w}, \tilde{w})} h(z, \bar{z}),$$
$$X'_w = \{z \in X \mid (\tilde{z}, \tilde{w}) \neq 0\},$$

where w ranges over $CP(N)$. Hence $h_w(z)$ is connected with the distance $\delta(z, w)$ by the relation: $\cos\delta(z, w) = | h_w(z) |$ (see §7). At this stage, the following proposition is almost clear.

Proposition. *h_w is a Morse function on X'_w for generic w.*

Using Bertini's theorem, we can get

Corollary. *There exists at least one $w \in CP(N)$ such that $X'B_w$ is a nonsingular subvariety and h_w is a Morse function.*

9. An application of the Morse theory

Here in this section, we owe [11] very much. By means of the corollary in the preceding section we can find a continuous (real positive) function on X such that $h \mid X - X \cap H$ is a Morse function where H is a hyperplane with nonsingular $X \cap H$. Note that h assume the value 0 on $X \cap H$, and define $X_a = h^{-1}(a, +\infty)(a > 0)$. Then for sufficiently small ε, X_ε is contained in a tubular neighborhood (in X) of $X \cap H$. Since the Euler characteristic $\chi(\ )$ is additive, we have $\chi(X) = \chi(X, X_\varepsilon) + \chi(X_\varepsilon, \varnothing)$. On the other hand, $\chi(X_\varepsilon, \varnothing) = \chi(H \cap H)$ because $X \cap H$ is a deformation retract of X_ε. Hence we have

$$\chi(X) = \chi(X, X_\varepsilon) + \chi(X \cap H).$$

Suppose that h have exactly k critical points with indices $r_1, \cdots, r_k$ respectively, in $X - X \cap H$. Then X has the same homotopy type as $X_\varepsilon \cup \sigma_1 \cup \cdots \cup \sigma_k$ where σ_i are r_i-cells $(i = 1, \cdots, k)$. Write

$$\begin{aligned}\alpha &= \begin{array}{l}\text{the number of critical points with positive indices of } h \mid X - X \cap H \\ -\text{the number of critical points with negative indices}\end{array} \\ &\qquad \text{(by the Morse theory)} \\ &= \begin{array}{l}\text{the number of even-dimensional cells } \sigma_i - \text{the number of} \\ \text{odd-dimensional cells } \sigma_i, = \chi(X, X_\varepsilon).\end{array}\end{aligned}$$

Then

$$\begin{aligned}\alpha &= \text{the number of geodesic perpendiculars from a generic point of } CP(N)\\ &= \text{degree of the Gauss map } G_\phi\\ &= \int_X c_n(T(X)\otimes[-H]).\end{aligned}$$

We can therefore state our final formula

$$\chi(X) = \chi(X\cap H) + \int_X c_n(T(X)\otimes[-H]).$$

10. A formula on Chern classes

Let $I(X)$ be the homogeneous ideal of X. For $f \in I(X)$ we denote by $df(\tilde{z})$ the linear form on $\mathbf{C}^{N+1}$ defined by

$$df(z)(w_0,\cdots,w_N) = \sum_A \left(\frac{\partial}{\partial z_A} f\right) w_A.$$

The subspace of $(\mathbf{C}^{N+1})^*$ which is spanned by $df(\tilde{z})$ $(f \in I(X))$ is determined by z where $z = \pi(\tilde{z})$. Hence we denote it by $\mathcal{S}_z$. Identifying the variety of $(N-n)$-planes in $(\mathbf{C}^{N+1})^*$ with $G(N-n, N+1)$, we have a map: $X \to G(N-n, N+1)$ which sends z to $\mathcal{S}_z$. We denote by $\mathcal{S}$ the vector bundle induced from the tautological vector bundle over $G(N-n, N+1)$. On the other hand, we can assign to each $z \in X$ the linear subspace

$$\{(w_0,\cdots,w_N) \mid df(\tilde{z})(w_0,\cdots,w_N) = 0 \quad \text{for any } f \in I(X)\}.$$

This gives rise to a map: $X \to G(n+1, N+1)$, which induces a vector bundle T over X from the tautological vector bundle over $G(n+1, N+1)$. Let us consider the product bundle over X with typical fiber $\mathbf{C}^{N+1}$. We denote it by $\mathcal{C}^{N+1}$. To each $(\tilde{z}, \tilde{w}) \in \mathcal{C}^{N+1}$ we can assign a linear form on $(\mathcal{S}_z)^*$ by defining

$$\kappa((\tilde{z}, \tilde{w}))df(\tilde{z}) = df(\tilde{z})(w_0,\cdots,w_N).$$

Note that the right side defines the same element for different $\tilde{z}$ over z in $\mathcal{S}^*$, the dual of S. Thus we can find an exact sequence of vector bundles over X

$$0 \to \mathcal{T} \to \mathcal{C}^{N+1} \xrightarrow{\kappa} \mathcal{S}^* \to 0$$

where $\mathcal{T}$ is the kernel of κ.

Define the action of the multiplicative group $\mathbf{C}^*$ by $\lambda(\tilde{z}, \xi) = (\lambda\tilde{z}, \lambda\xi)$, where $\lambda \in \mathbf{C}^*$ and $(\tilde{z}, \xi) \in T(\mathbf{C}^{N+1} - 0)$. Taking the quotient by this action, we have a vector bundle homomorphism of $[H] + \cdots + [H]$ $(N+1$ copies) to

$T(CP(N))$, where $[H]$ denotes the hyperplane bundle over $CP(N)$. This homomorphism can be imbedded in an exact sequence called the Euler sequence:

$$0 \to \mathcal{C} \to [H] + \cdots + [H] \to T(CP(N)) \to 0$$

where $\mathcal{C}$ is the product bundle over $CP(N)$ with typical fiber $\mathbf{C}$, [7]. Quite analogously to the Euler sequence over the complex projective space, we have

$$0 \to [-H] \to \mathfrak{T} \to T(X) \otimes [-H] \to 0$$

or

$$0 \to \mathcal{C} \to \mathfrak{T} \otimes [H] \to T(X) \to 0.$$

These exact sequences imply two relations among the total Chern classes:

$$c(\mathcal{S}^*)c(\mathfrak{T}) = 1,$$
$$C(T(X) \otimes [-H])c([-H]) = c(\mathfrak{T}).$$

From these we obtain a formula:

$$c(T(X) \otimes [-H])c([-H])c(\mathcal{S}^*) = 1, \tag{13}$$

using which we may calculate $c_n(T(X) \otimes [-H])$ in some cases.

Now suppose X to be a complete intersection. Then we can find $N - n$ homogeneous polynomials $f_1, \cdots, f_{N-n}$ which generate $I(X)$. We write

$d_1 + 1 =$ the degree of $f_1, \cdots, d_{N-n} + 1 =$ the degree of f_{N-n},

$d =$ the multi-degree, i.e., $= d_1 + \cdots + d_{N-n} + (N - n)$.

In this case we have

$$\mathcal{S} \simeq [-H]^{d_1} + \cdots + [-H]^{d_{N-n}},$$

where the exponents d_l mean the d_l-fold tensor product with itself. The formula (13) therefore turns out to be

$$c(T(X) \otimes [-H])c([-H])c(H^{d_1}) \cdots c(H^{d_{N-n}}) = 1, \tag{14}$$

which allows us to compute $c_n(T(X) \otimes [-H])$. In fact, write

$$c(H) = 1 + \omega.$$

Then we have

$$c(T(X) \otimes [-H]) = \frac{1}{(1 - \omega)(1 + d_1\omega) \cdots (1 + d_{N-n}\omega)}, \quad (\mathrm{mod}\ \omega^{n+1}). \tag{15}$$

12. Examples

1. Suppose X to be a linear subspace. This is the simplest example. From (15) we see

$$c_n(T(X) \otimes [-H]) = \text{the } n\text{th term of } \frac{1}{(1-\omega)} = \omega^n.$$

Hence

$$n(X) = \int \omega^n = 1.$$

2. Take a nonsingular plane curve of degree d as the next example. Obviously $\chi(X \cap H) =$ the degree of $X = d$. On the other hand from (15) we have

$$c_1(T(X) \otimes [-H]) = -(d-2)\omega,$$

and therefore

$$n(X) = -(d-2)\int \omega = -d(d-2),$$

$$\chi(X) = -d(d-3).$$

The latter is the so-called genus formula, [8], [9]. Since

$$c_1(T(X) \otimes [-H]) = c_1(X) - \omega,$$

we can get the Gauss-Bonnet formula

$$\int c_1(X) = \chi(X) - \chi(X \cap H) + \int \omega = \chi(X).$$

3. The final example is the complex quadric defined by

$$z_0^2 + \cdots + z_N^2 = 0.$$

In this case we know that the Betti numbers $b_0, \cdots, b_n$ of X are given by

$$b_{2i-1} = 0,\ b_{2i} = 1 \text{ unless } 2i = n,$$
$$b_n = 2 \text{ for } n \equiv 0 \pmod 2,\ b_0 = 1,$$

where $i = 1, \cdots, n$, and of course $n = N - 1$, so that

$$\chi(X) = \begin{cases} n+1 \text{ if } n \equiv 1 \pmod 2, \\ n+2 \text{ if } n \equiv 0 \pmod 2. \end{cases}$$

We therefore have

$$n(X) = \begin{cases} 0 \quad \text{if } n \equiv 1 \pmod 2, \\ 2 = \text{degree of } X \text{ if } n \equiv 0 \pmod 2. \end{cases} \tag{16}$$

But to get (16), it is easier to use the integral of c_n. Actually the nth Chern class is given by the nth term of the series

$$\frac{1}{1-\omega^2} = 1 + \omega^2 + \omega^4 + \cdots .$$

This implies

$$c_n(T(X) \otimes [-H]) = \begin{cases} 0 & \text{if } n \equiv 1 \pmod 2, \\ \omega^n & \text{if } n \equiv 0 \pmod 2. \end{cases}$$

Hence we can obtain the same result as (16).

Now we know that the jacobian of G_Φ is always nonnegative for the even-dimensional complex quadrics (§3). Therefore we have the following theorem.

Theorem. *We can draw exactly two geodesic perpendiculars from a generic point of $CP(N)$ to an even-dimensional complex quadric.*

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